Abstract.
In the three-period overlapping generation model [1958] we prove the existence of a samuelsonian conjecture about the possible existence of positive younger agents saving under the assumption of heterogeneous agents. In the model presented here agents are heterogeneous in terms of their preferences. The samuelsonian positive young agent’s saving conjecture has two consequences when we consider the market’s transactions rules. First, we examine the case when a part of the younger agents lend to other young agents. The lenders act like financial intermediaries and the price of credit is equal to the asset’s fundamental values. But the fact that the younger agents can have positive savings leads us to consider the case when they lend to the older agents when transaction market transactions are anonymous. All agents may borrow and we then find a difference between the fundamental value and the market price values. Therefore we prove the existence of endogenous bubbles under the anonymous market transaction rule.

Keywords: Endogenous Assets Bubbles, Three-Period Overlapping Generation Model, Heterogeneous Agents.

1. Introduction
The study about assets bubbles is related to the assets pricing valuation. In a seminar contribution, Blanchard-Watson [1982] mentioned that the market price and the market fundamental value of an asset can be different in a partial equilibrium setup; the difference is defined as rational bubble “bullerationelle”. In the OLG general equilibrium model, Tirole [1985] also defined bubbles as the difference between the market price and market fundamental of any assets; in his model he considers the allocation between productive investment and assets bubbles. We can also find some argument in Hahn [1966] and Shell-Stiglitz [1967] for the model with the long-lived agents, the agents can make a profit by
doing speculating investment and the bubble could always existing. The question now is can asset bubbles exist in pure exchange economic without production?

We try to answer the question by using the well-known Samuelson [1958] three-period overlapping generation model without production. In this model each agent lives three periods. In each period there are three representative agents: the young agent, the middle-aged agent, and the old agent. To fulfill the consumptions in the retired period, the old agents need to receive goods from the younger agents, since the perishable commodity can’t be saved over time. In the commemorative book “Samuelsonian Economics and the Twenty-First Century” Solow [2006] points out the central property of negative young agents saving. That’s why in the model, the non-autarky equilibrium is characterized by the consumption and loan between different generations. Despite this property, Samuelson remarks that the young agent saving is always negative in free market.

Balasko’s contribution The Overlapping-Generation Model III [1981] also shows that if the overlapping-generation model has log-linear utility function by using the transversality theory the equilibrium of competitive market always satisfy and the price always has same value with the market fundamental. In Symposium on Bubble’s paper Stiglitz [1990] made a resume from some economist view in how to define the asset fundamental value, in some competitive market the fundamental value cannot justify the market price due some factor. Stiglitz also mention that the transversality condition always satisfy, its depend on the market condition whether the asset price will be the same with the fundamental value or not as long as the agents has an infinite planning horizon.

Our main purpose of this paper is to determine the conditions of the existence bubbles in the samuelsonian three-period overlapping generation model. With the assumption of the agents preferences heterogeneity it’s possible that the part of young population become lender, and under anonymous rules of free market we show the existence of endogenous bubble when old agents can borrow to young agents through the market. In our model we proved that the anonymous transaction can involve the existence of asset bubble because the possibility of unpaid debt which was borrows by the old agents. The bubble can cause the younger agents who life after need to pay more and more.

In the first part we modify agent’s preference pattern and examine the equilibrium characteristic. In the second part we introduce agent’s preference heterogeneity. And in the last part we study two type equilibrium conditions: equilibrium with financial intermediation.
to get the fundamental values of asset and equilibrium in anonymous market to obtain the market bubbleprice.

2. The Model of Homogeneous Agents with Linear Utility Function in Consumption Pattern

The samuelsonian three-period model is based on the logarithmic utility function with symmetric pattern. The production of the perishable consumption good is simple: each young and middle age produce one unit. In this set up the author shows that the young agents saving are always negative. The young agents always consume more than their current income. Therefore they will borrow quantities of goods from the middle-aged generation. Meanwhile, the middle-aged lend their goods to the young because they expect to receive the payment on the next period. The reimbursement ensures the agent’s consumption in their third retired period of life. Moreover in the three-period samuelsonian scheme the assets price value in competitive market is always equal to its fundamental values (i.e. the reimbursement value).

Beside the above condition there is a suggestion from samuelson’s colleague “if the model end up with positive saving of young agents then the voluntary trade can be happened because not like the middle-aged the young willing to trade with anyone”. Moreover, Kranton [1996] assumes that agents will choose the anonymous transaction or reciprocal but not both in samuelsonian model we can see clearly that the middle-aged choose personal trade. If we can build the three-period OLG model, with positive young agents saving, then the conjecture of samuelson’s colleague can possibly happen because the young agent will possibly choose the anonymous transaction.

In this study we present some modifications of the basic set-up to examine the role of the utility function’s concavity on the existence of asset bubbles.

\[ U = \sum u_i, \text{ with } i = 1, ... 3. \]

For the sake of simplicity we use a mix of quasi-concave and linear utility function. For instance we define an individual utility with a linear pattern in first period and logarithmic pattern for the second and the last periods:

\[ U(c_1, c_2, c_3) = \alpha \times c_1 + \log c_2 + \log c_3, \]

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1 The reader can find an interesting presentation of the three period model in De la Croix and Michel [2005].
where $\alpha$ is the constant of linear consumption function, that defined the agents consumption preference.

We assume that agents are heterogeneous in terms of their preferences in their first life’s period. They produce one unit of the perishable goods (e.g. chocolates) in the first and second period of their lifetime. For the budget constraint each agent life’s consumption must meet the discounted values of their production. Thus for the first modification in utility function we have the maximization problem as:

$$\max \{ \alpha \times c_1 + \log c_2 + \log c_3 \},$$

subject to the intertemporal budget constraint:

$$c_1 + \beta_t \times c_2 + \beta_{t+1} \times c_3 = 1 + 1 \times \beta_t,$$

with $\beta_t = \frac{1}{1+r_t} = \frac{p_t}{p_{t+1}}$ as the discounted rate between goods of period $t$ traded for the goods in $t+1$. Let $p_t$ be the price of goods in period $t$, we $\beta_t$ can also define as the price perishable goods today expressed in term of tomorrow’s price then $\beta_t$ and $\beta_{t+1}$ must be positive.

From the above maximization problem, the optimal consumption and saving functions are:

$$s_1(\beta_t, \beta_{t+1}) = 1 - c_1(\beta_t, \beta_{t+1}) = 1 - (\beta_t + 1 - \frac{2}{\alpha}) = \frac{2}{\alpha} - \beta_t,$$

$$s_2(\beta_t, \beta_{t+1}) = 1 - c_2(\beta_t, \beta_{t+1}) = 1 - \frac{1}{\alpha \times \beta_t},$$

$$s_3(\beta_t, \beta_{t+1}) = 0 - c_3(\beta_t, \beta_{t+1}) = -\frac{1}{\alpha \times \beta_t \times \beta_{t+1}}.$$

The sum of saving of each agent must be zero in every period when the goods can’t be saved over time. The competitive market equilibrium must fulfill the following condition:

$$s_3(\beta_{t-2}, \beta_{t-1}) + s_2(\beta_{t-1}, \beta_t) + s_1(\beta_t, \beta_{t+1}) = 0.$$

Using the equilibrium condition (4) we get the recursive form of discounted rate factor $\beta_t$ as:

$$\beta_t = \left(1 + \frac{2}{\alpha}\right) - \frac{1}{\alpha \times \beta_{t-1} \times \beta_{t-2}} - \frac{1}{\alpha \times \beta_{t-1}}.$$

With the initial values of discounted rate $\beta_t$ are being calculated separately. We know that after some periods of time the value of $\beta_t$ will be convergent to $x$. By substituting $\beta_{t+2} = \beta_{t+1} = \beta_t = x$ into the equation (5), we obtain the cubic equation:

$$(x - 1) \left(x^2 - \frac{2}{\alpha} x - \frac{1}{\alpha}\right) = 0.$$
Therefore we conclude that after several periods, $\beta_t$ will converge to $\beta$. We can define $\beta$ as a function of parameter $\alpha$:

$$\beta = x = g(\alpha) = \frac{1}{\alpha} + \frac{1}{\alpha} \sqrt{1 + \alpha}, \text{ for } \alpha \in [0, \infty).$$

(7)

Figure 1. Optimal values of discounted rates in competitive market $\beta = g(\alpha)$ for utility function (1)

Figure 1 shows us that in interval $0 < \alpha < \infty$ the function $\beta(\alpha)$ is monotonically decreasing function; the values of $\beta$ is decreasing when we choose higher values of parameter. From cubic equation (6) we have the roots of the quadratic equation which are:

$$x = \frac{1}{\alpha} + \frac{1}{\alpha} \sqrt{1 + \alpha}, \alpha > 0; \text{ and } x = \frac{1}{\alpha} - \frac{1}{\alpha} \sqrt{1 + \alpha} \text{ as the negative roots for any } \alpha > 0.$$ 

We have to eliminate the negative roots because its economically meaningless.

Figure 2. Optimal values of interest rates $r(\alpha)$ for utility function (1)

From the definition we know that $\beta_t$ is the discounted rate between goods of period $t$ traded for the goods in $t + 1$, then we can also define $r$ as the function of $\alpha$. From figure 2 we can see that $r(\alpha)$ is monotonically increasing function. From both figure 1 and 2, if we choose $\alpha = 3$ we have $\beta(\alpha = 3) = 1$ and $r(\alpha = 3) = 0$. Thus can be concluded that
r(α = 3) = 0 is an important point, because the economy converge to the optimum biological condition. This result correspond to the remark of Samuelson [1958] “(...) for some special pattern of time preference, the competitive solution might coincide with biological optimum”. The log-linear utility function verifies the coincidence between the competitive equilibrium solution and biological optimum, as stated in the following proposition:

Proposition 1: for utility function with log-linear pattern: $U = \alpha \times c_1 + \log c_2 + \log c_3$. The competitive market solution equalsto the optimum biological equilibrium, if the preference parameter $\alpha = 3$.

From figure 2 we can also find that the model’s results can be divided into two intervals: Thenegative and positive free market interest rate that depend on the values of $\alpha$:

1. If we choose $\alpha$ from the interval $0 < \alpha \leq 3$, we have $r < 0$ negative interest rate.
2. And if we choose $\alpha$ from the interval $3 < \alpha < \infty$, then $r > 0$ positive interest rate.

After finding the discounted value for competitive market, we continue with analyzing the saving function. The discounted values of free market will convergent to $\beta = x = \frac{1}{\alpha} + \frac{1}{\alpha} \sqrt{1 + \alpha}$ for $\alpha \in [0, \infty)$. Then for $\beta_{t+1} = \beta_t = \beta$, we can clearly say that $s_2(\beta)$ always positive and $s_3(\beta)$ always negative, or all parameters $\alpha > 0$. As for young agents savings function:

$s_1(\beta(\alpha)) = \frac{2}{\alpha} - \beta(\alpha)$;

$s_1(\alpha) = \frac{2}{\alpha} - \left(\frac{1}{\alpha} + \frac{1}{\alpha} \sqrt{1 + \alpha}\right) < 0$, for $\alpha \in [0, \infty)$.

From above property we can conclude that $s_1(\alpha)$ is increasing function, with $\lim_{\alpha \to \infty} s_1(\alpha) = 0$.

The market equilibrium being reach is explained in the proposition 2.

Proposition 2: for with log-linear pattern: $U = \alpha \times c_1 + \log c_2 + \log c_3$, with equilibrium economic condition: $s_3(\beta_{t-2}, \beta_{t-1}) + s_2(\beta_{t-1}, \beta_t) + s_1(\beta_t, \beta_{t+1}) = 0$ the transversality condition is verified.

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1 In this paper we assume that all models are defined under the stationary population hypothesis (the growth rate is zero).
We prove Proposition 2 first by defining $d_t$ as the amount of young agent’s debt, with middle-agent as the lender in period $t$ then: $d_t = 1 - s_2(\beta_{t-1}, \beta_t)$.

When $\beta_t$ approach the optimal value $\beta$, we substitute $\beta_t = \beta_{t+1} = \beta_{t+2} = \beta = \frac{1}{(1+r)}$ into the equilibrium condition (4), then we will see that the consumption of old agents will be satisfied by the refund of the debt with the interest rate:

$$(1 + r_t) \times d_t = (1 + r_t) \left( \frac{1}{\alpha \times \beta_{t-1}} \right) = (1 + r) \left( \frac{1}{\alpha \times \beta} \right) = \frac{1}{\alpha \times \beta} = c_3(\beta),$$

with $\beta_{t+1} = \beta_{t+2} = \beta$

Therefore it can be concluded that all the transactions are the personal trade; the debts path is sustainable. The agents always able to pay their debt at the market interest.

The result of the log-linear utility function attains the same output as the samuelsonian three-period model where there is no anonymous transaction existing. In addition, we proved that with the modification in first consumption the solution for competitive market can meet the biological optimum condition solution (Proposition 1), with the assumption of stationary population. We also proved that, in homogeneous model with the asymmetric log-linear utility function the increasing of interest rate will cause the young agents consume less, but the values still more than one unit commodity:

$$c_1(\beta) = 1 + \frac{1}{\alpha} + \frac{1}{\alpha} \sqrt{1 + \alpha} > 0, \text{for } \alpha \in [0, \infty).$$

As a consequence, young agents saving will also increasing but the values still negative so that the anonymous transaction still not possible to happened in the competitive market.

3. Overlapping Generation Model with Heterogeneous Agents

It had been proved in the previous section that in homogeneous model with log-linear utility function the young agents saving will increase along with the increasing of interest rate but the anonymous transaction still not existing. Now, if there are two types of agents in one population what will happen in competitive market? In this section we develop the model with heterogeneous agents, in which we assume that there are two types of agents who have different consumption preference.

Next if there are young agents who have positive saving will they choose to do the anonymous transaction or not and what will happened in economic. In Fafchamps [2011] and Kimbrough et al. [2008] it shows that the transition from personalized to anonymous
exchange might increase the market value or decrease the market values depend on the situation. The situation where the market decreasing can happen and the agents become poorer if the anonymous exchange that happened in rural communities where the agents only concentrate in one exchange modality Kumar and Matsusaka [2009]. From the result of previous study the anonymous transaction can lead to both good and bad effect in economic such as bubbles.

In this heterogeneous model the first group has the logarithmic utility function: \( U = \log c_1 + \log c_2 + \log c_3 \) like the samuelsonian model. The second group consist of agents who have the log-linear utility function as: \( U = \alpha \times c_1' + \log c_2' + \log c_3' \). We maximize each utility functions separately for each type of agent. First we calculate separately the saving function for each agent, maximize each \( U_1 \) and \( U_2 \):

\[
U_1 = \log c_1 + \log c_2 + \log c_3, \quad (8)
\]

\[
U_2 = \alpha \times c_1' + \log c_2' + \log c_3',
\]

subject to budget constraint:

\[
c_1 + c_2 \times \beta_t + c_3 \times \beta_t \times \beta_{t+1} = 1 + 1 \times \beta_t.
\]

For each \( U_1 \) and \( U_2 \) the consumption pattern and saving function are:

\[
c_1(\beta_t, \beta_{t+1}) = \frac{1}{3} + \frac{\beta_t}{3}; \quad s_1(\beta_t, \beta_{t+1}) = \frac{2}{3} - \frac{\beta_t}{3};
\]

\[
c_2(\beta_t, \beta_{t+1}) = \frac{1}{3} + \frac{1}{3 \times \beta_t}; \quad s_2(\beta_t, \beta_{t+1}) = \frac{2}{3} - \frac{1}{3 \times \beta_t};
\]

\[
c_3(\beta_t, \beta_{t+1}) = \frac{1}{3 \times \beta_t \times \beta_{t+1}} + \frac{1}{3 \times \beta_t \times \beta_{t+1}}; \quad s_3(\beta_t, \beta_{t+1})
\]

\[
= \frac{1}{3 \times \beta_t \times \beta_{t+1}} - \frac{1}{3 \times \beta_t \times \beta_{t+1}};
\]

\[
c_1'(\beta_t, \beta_{t+1}; \alpha) = \beta_t + 1 - \frac{2}{\alpha}; \quad s_1'(\beta_t, \beta_{t+1}; \alpha) = \frac{2}{\alpha} - \beta_t;
\]

\[
c_2'(\beta_t, \beta_{t+1}; \alpha) = \frac{1}{\alpha \times \beta_t}; \quad s_2'(\beta_t, \beta_{t+1}; \alpha) = 1 - \frac{1}{\alpha \times \beta_t};
\]

\[
c_3'(\beta_t, \beta_{t+1}; \alpha) = \frac{1}{\alpha \times \beta_t \times \beta_{t+1}}; \quad s_3'(\beta_t, \beta_{t+1}; \alpha) = -\frac{1}{\alpha \times \beta_t \times \beta_{t+1}},
\]

where \( c_i \) and \( s_i \) are the consumption pattern and saving function that correspondent with \( U_1 \), while \( c_i' \) and \( s_i' \) correspondent with \( U_2 \). From the saving functions (9) it can be resume that the saving functions values will depend on the market interest rate and parameter \( \alpha \) as the constant of linear consumption function.
Since in the model the perishable commodity can’t be saved over time, the sum of saving of each agent must be zero in every period. In population with heterogeneous agents we have equilibrium condition as:

$$y \left[s_1(\beta_t, \beta_{t+1}) + s_2(\beta_{t-1}, \beta_t) + s_3(\beta_{t-1}, \beta_{t-2})\right] +$$

$$(1-\gamma)[s'_1(\beta_t, \beta_{t+1}) + s'_2(\beta_{t-1}, \beta_t) + s'_3(\beta_{t-1}, \beta_{t-2})] = 0, \quad 0 < \gamma < 1,$$

with $y$ represents the agent proportion. Using the equilibrium condition (10), the recursive forms of $\beta_t$ that satisfied the heterogeneous model (8) is:

$$\left(1 - \frac{2}{3} \gamma\right) \beta_t = \left(\frac{y}{3} + 1 + \frac{2(1-\gamma)}{\alpha}\right) - \left(\frac{2y}{3} + \frac{(1-\gamma)}{\alpha}\right) \frac{1}{\beta_{t-1}} - \left(\frac{y}{3} + \frac{(1-\gamma)}{\alpha}\right) \frac{1}{\beta_{t-1} \beta_{t-2}},$$

after some periods the values $\beta_t$ will converge to roots of cubic equation:

$$\left(1 - \frac{2}{3} \gamma\right)x^3 - \left(\frac{y}{3} + 1 + \frac{2(1-\gamma)}{\alpha}\right)x^2 + \left(\frac{2y}{3} + \frac{(1-\gamma)}{\alpha}\right)x + \left(\frac{y}{3} + \frac{(1-\gamma)}{\alpha}\right) = 0.$$ 

Equation above has three roots:

$$x_1 = 1;$$

$$x_2 = \frac{\left(y + \frac{2(1-\gamma)}{\alpha}\right)^2 + 4\left(1 - \frac{2}{3} \gamma\right)\left(\frac{y}{3} + \frac{(1-\gamma)}{\alpha}\right)}{2\left(1 - \frac{2}{3} \gamma\right)};$$

$$x_3 = \frac{\left(y + \frac{2(1-\gamma)}{\alpha}\right)^2 - 4\left(1 - \frac{2}{3} \gamma\right)\left(\frac{y}{3} + \frac{(1-\gamma)}{\alpha}\right)}{2\left(1 - \frac{2}{3} \gamma\right)}.$$

We know that $\beta_{t-1} = \beta_t = \beta_{t+1} = \beta$ is one of the roots, as mentioned before $\beta$ must be positive so the optimal value of discounted factor is:

$$\beta = 1 \text{ or } \beta = \frac{\left(y + \frac{2(1-\gamma)}{\alpha}\right)^2 + 4\left(1 - \frac{2}{3} \gamma\right)\left(\frac{y}{3} + \frac{(1-\gamma)}{\alpha}\right)}{2\left(1 - \frac{2}{3} \gamma\right)}.$$ 

Discounted factor $\beta$ must also make the recursive form stable. So we can resume that the optimal value of discounted factor must satisfy the three properties of Criteria1:

1. $\beta > 0$;
2. $\beta$ must be real number because there is possibility that the cubic equation has complex roots;
3. $\beta$ must made the recursive form stable.\(^3\)

Now, we can write the savings functions (9) as the function of $\beta$, $s_i(\beta)$ and $s'_i(\beta; \alpha)$ for $i = 1, 2, 3$. Note that the middle-aged agents have different saving functions for each type of

\(^3\) We will always use the criteria for determine the values of discounted values for all models.
utility function. The middle-aged agents with symmetric utility function (samuelsonian model) have savings function, $s_2(\beta) = \frac{2}{3} - \frac{1}{3\beta}$, depend only on the market interest rate. Meanwhile, the savings of middle-aged with log-linear utility function, $s'_2 (\beta; \alpha) = 1 - \frac{1}{\alpha \times \beta}$, depends on the market interest rate and the first period consumption preference parameter. In this model the middle-aged decide also to lend their goods directly to the younger agents to insure their consumption in the retirement period, so the middle-aged must have positive saving:

$$s_2(\beta) = \frac{2}{3} - \frac{1}{3\beta} > 0 \text{ and } s'_2 (\beta; \alpha) = 1 - \frac{1}{\alpha \times \beta} > 0,$$

under the following conditions:

$$\beta > 0.5 \text{ and } \alpha \times \beta > 1.$$  \hspace{1cm} (16)

Now we consider the pattern of young agents saving function. In the heterogeneous case (8) the savings of young agents also have the same pattern as the middle-aged. The young agents savings function with symmetric logarithmic utility function depends only on the market interest rate, and the savings of young agents with log-linear utility function depends on the market interest rate and the first period consumption preference parameter:

$$s_1(\beta) = \frac{2}{3} - \frac{\beta}{3}; s'_1(\beta; \alpha) = \frac{2}{\alpha} - \beta.$$  \hspace{1cm} (18)

Using the saving function (18) and the positivity conditions (17) the young agents savings are positive under the two following cases:

1. $s_1(\beta) > 0$ if $0.5 < \beta < 2$ and $\alpha \times \beta > 1$;
2. or $s'_1 (\beta; \alpha) > 0$ if $1 < \alpha \times \beta < 2$.

Proposition 3. In the heterogeneous agent model where agents have the following different utilities functions: $U_1 = \log c_1 + \log c_2 + \log c_3$ and $U_2 = \alpha \times c'_1 + \log c'_2 + \log c'_3$. The savings function with the log utility is positive, $s_1(\beta) > 0$, when $0.5 < \beta < 2$ and $\alpha \times \beta > 1$. In the case of log-linear utility the saving function is positive, $s'_1(\beta; \alpha) > 0$, when have $1 < \alpha \times \beta < 2$.

We know that in samuesonian OLG model the young agents saving function is always negative. In heterogeneous agents model we proved that the young agents can have positive saving when the interest rate is negative. When the young agents have positive saving there are two possibilities in the competitive market. First young agents with positive saving will choose to lend their perishable goods directly to others young agents, with negative saving,
following equilibrium condition (10). In this case transactions are personal (Kranton [1996]) and the young agents who lend their goods act like financial intermediaries. In second case, the young agents with positive saving make anonymous transaction in the competitive market. This transaction process can allow the old agents to borrow from market without payment constraint. Now we will see if the asset price still represents the fundamental values or not?

3.1. The fundamental values with heterogeneous agents

We have proved that in heterogeneous agents model there always a group of young agents who can have positive saving if we choose specific values of parameters $\alpha$ and $\gamma$ (Proposition 3). In this model we have two types of agents, agents type-1 is the group of agents with positive saving when they young and type-2 is the group with negative young agents savings (figure 3 and 4).

![Figure 3. Market scheme when young agents-1 have positive saving in heterogeneous agents model](image)

To calculate the fundamental values, the equilibrium condition is determined under the situation where the agents who borrow some goods must refund their debt with the interest rate in the next period equilibrium condition (10) and illustrated by figure 3. From the previous result the optimal value of discounted factor in competitive market that follow the equilibrium condition (10) is $\beta = 1$ or $\beta = \frac{(\gamma + \frac{2(1-\gamma)}{a}) + \sqrt{(\gamma + \frac{2(1-\gamma)}{a})^2 + 4(1-\frac{\gamma}{2})(\gamma + \frac{1-\gamma}{a})}}{2(1-\frac{\gamma}{2})}$, and satisfy the three properties of (Criteria 1), this discounted factor will define the fundamental value of our perishable goods. From the definition of discounted rates $\beta_t = \frac{1}{1+r_t} = \frac{p_t}{p_{t+1}}$, we
can calculate the fundamental values for our heterogeneous model. Write $p_t$ as the fundamental value then:

$$p_t(\beta) = p_{t+1} \times \beta.$$ 

Let $s_1(\beta(\alpha, \gamma))$ negative (type-2) and $s_1^*(\beta(\alpha, \gamma))$ positive (type-1), the debt of young agent type-2 is: $\gamma \times d_t = -\gamma \times s_1(\beta(\alpha, \gamma))$. This debt is obtained not only from the middle-aged agents with same consumption but also from the different types young agents:

$$-\gamma \times d_t = \gamma \times c_2(\beta(\alpha, \gamma)) + (1 - \gamma) \left[ c_2^*(\beta(\alpha, \gamma)) - (-s_1^*(\beta(\alpha, \gamma))) \right].$$

So we can resume the situation in the proposition 4:

Proposition 4: In the heterogeneous agent model, with two type of agents who have utility function as: $U_1 = \log c_1 + \log c_2 + \log c_3$ and $U_2 = \alpha \times c_1' + \log c_2' + \log c_3'$, under the equilibrium condition (10), when there is a group of young agents who has positive saving (type-1), that group of agents play the role as financial intermediation to keep the equilibrium condition satisfy.

We consider now, the market’s price evaluation of the perishable goods trade among agents where anonymous transaction are authorized (the old agents can borrow from the market).

3.2. The market price and the existence of endogenous bubbles under anonymous rules of free market

Under anonymous rules of free market, when there are young agents who have positive saving (type-1) they will choose the anonymous transaction then the old agents can borrow through the market. This situation triggers bubbles because there is a difference between the market price and market fundamental value of the goods. To determine the market price we use the market equilibrium condition:

$$\gamma [s_1(\beta_t^*, \beta_{t+1}^*) + s_2(\beta_{t-1}^*, \beta_t^*) + s_3(\beta_{t-2}^*, \beta_{t-1}^*)] + (1 - \gamma) [s_1^*(\beta_t^*, \beta_{t+1}^*) + s_2^*(\beta_{t-1}^*, \beta_t^*) + s_3^*(\beta_{t-2}^*, \beta_{t-1}^*)] - b = 0,$$

(19)
with variable \( b \) as the fraction of the debt that will not be refund, \( 0 < b < 1 \). We will show if there is the debt that will not be refund then the market interest rate will be higher that the market interest rate that define by equilibrium condition (19).

![Diagram](image)

**Figure 4. Anonymous Transaction in Competitive Market**

When we use the condition of equilibrium with heterogeneous agents (19), we establish the recursive equation of \( \beta_t^* \) (e.g. (8)):

\[
\left(1 - \frac{2}{3}\alpha\right) \beta_t^* = \left(\frac{2\gamma}{3} + 1 + \frac{2(1-\gamma)}{\alpha} - b\right) - \left(\frac{2\gamma}{3} + \frac{(1-\gamma)}{\alpha}\right) \frac{1}{\beta_{t-1}^*} - \left(\frac{2\gamma}{3} + \frac{(1-\gamma)}{\alpha}\right) \frac{1}{\beta_{t-1}^* \beta_{t-2}^*},
\]

(20)

We assume that \( \beta_t^* \) will converge to \( x \) and we set \( \beta_{t+2}^* = \beta_{t+1}^* = \beta_t^* = x \), so we can write the recursive formula above as the cubic equation:

\[
\left(1 - \frac{2}{3}\alpha\right) x^3 - \left(\frac{2\gamma}{3} + 1 + \frac{2(1-\gamma)}{\alpha} - b\right) x^2 + \left(\frac{2\gamma}{3} + \frac{(1-\gamma)}{\alpha}\right) x + \left(\frac{2\gamma}{3} + \frac{(1-\gamma)}{\alpha}\right) = 0,
\]

(21)

using the cubic equation (21) we can find the value \( x \). We can define the roots of (21) as a function of \((\alpha, \gamma, b) : \beta^*(\alpha, \gamma, b)\), with \(0 < b < 1\). For the sake of simplicity we choose specific values of the parameters \( \alpha \) and \( \gamma \) that ensures \( s_1 \) or \( s_1^* \) positive, and write \( \beta^* \) as a function of \( b \):

\[
\beta^*(b), \text{ with } 0 < b < 1.
\]

(22)

We know that cubic equation (21) has three roots. We write the set of roots as \( \{a_1, a_2, a_3\} \), and we must choose \( \beta^* = a \) from \( \{a_1, a_2, a_3\} \) which satisfy all the condition of criteria 1. Now the optimum variable \( \beta^* \) is the discounted values that define the market price.
Then we have $\beta_{t+2} = \beta_{t+1} = \beta^*_t = x = a$, from the definition of discounted rates we can write:

$$\beta^*_t = r_t - d,$$

and the recursive form (20) as:

$$r_{t+2} = k \times r_{t+1} + l \times r_t + m,$$

with $k = \frac{(-d^2-k_1d)}{d^2}, l = \frac{(-d^2-k_1d-l_1)}{d^2}$, and $m = \frac{(k_1d^2+d_1l_1-m_1)}{d^2}$.

Let $A = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$, the transition matrix of recursive equation (20). The recursive equation (20) is stable if and only if all the eigenvalues of transition matrix $A$ has absolute value less than 1. So we have to choose $\beta^*(b) = a$ with $a \in \{a_1,a_2,a_3\}$ and $a$ must generate all the eigenvalues of transition matrix $A$ has absolute value less than 1. We can write cubic equation (21) as:

$$x^* - 1 - \left(1 - \frac{2}{3}\gamma\right)x^* - \left(\gamma + \frac{2(1-\gamma)}{\alpha} - b\right)x^* - \left(\frac{\gamma}{3} + \frac{1-\gamma}{\alpha} - b\right)\right) = 0,$$

with $\beta^*(b)$ as the roots of cubic equation (25), we must have $s_1(\beta^*(b)) > 0$ or $s_1(\beta^*(b)) > 0$. We can find the maximal value of $\beta^*(b)$ when $b = 0$, and we show that $\beta^*(b)$ is monotonically decreasing function in the interval $0 \leq b < b < 1$, where $b$ is the boundary value of $b$, for specific values of $\alpha$ and $0 < \gamma < 1$ that produce $s_1(\beta^*(b)) > 0$ or $s_1(\beta^*(b)) > 0$.

**Proposition 5:** In the heterogeneous agents model under anonymous rules of free market, if we choose specific values of $\alpha$ and $0 < \gamma < 1$ that produces $s_1(\beta^*(b)) > 0$ or $s_1(\beta^*(b)) > 0$, we will have the discounted values of free market convergent to $\beta^*$, with $\beta^*$ is the monotonically decreasing function of $b$. We proved that the endogenous bubble does exist under this situation.
The existence of bubble function (Proposition 5) is proved by the presence of the inverse function $\beta^* = f(b)$. $\beta^*$ can be expressed as $\beta^* = f(b)$ and we have proved that $f(b) = \beta^*$ is monotonically decreasing function in $0 \leq b < \bar{b} < 1$, so $f(b) = \beta^*$ is one-to-one function. This property guarantees that in interval $0 \leq b < \bar{b} < 1$ the inverse function $f(b) = \beta^*$ does exist. Let $g(\beta^*) = b$ is the inverse function with $b$ as the fraction of the debt that will not be refunded, we prove that the bubbles does exist under anonymous rule of free market in heterogeneous population. Furthermore we can define $\beta^*$ as the discounted factor of price value under competitive market with bubble.

From the definition of discounted values: $\beta_t = \frac{1}{1 + r_t} = \frac{p_t}{p_{t+1}}$, the fundamental values of heterogeneous model is defined by:

$$p_t(\beta) = p_{t+1} \times \beta,$$

(26)

in the condition where all the agents enter the free market under anonymous transaction of free market, the price value of commodity is defined by:

$$p_t(\beta) = p_{t+1} \times \beta^*,$$

with $\alpha$ and $0 < \gamma < 1$ that produces $1(\beta^*(b)) > 0$, or $s_1(\beta^*(b)) > 0$.  

(27)

Because $\beta^*(b)$ is monotonically decreasing function and $\beta^*(b = 0)$ is maximal value of $\beta^*(b) for 0 \leq b < 1 then \beta > \beta^*$. 

**Proposition 6**: Under anonymous rules of free market, the price value of the commodity is bigger than its fundamental value, because we have $\beta^* < \beta$ then we get $p_t(\beta^*) > p_t(\beta)$.

<table>
<thead>
<tr>
<th>b</th>
<th>Interest rate</th>
<th>Fundamental value</th>
<th>Price value</th>
<th>Bubble</th>
</tr>
</thead>
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<td>100</td>
<td>103.7075447</td>
<td>3.70754472</td>
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<td>100</td>
<td>103.9717197</td>
<td>3.97171969</td>
</tr>
</tbody>
</table>

Table 1. Fundamental Value, Asset Price, and Asset Bubbles in Three-period OLG Heterogeneous Agents Model for case $s_1 > 0$ and $s_1 < 0$
The Proposition 6 describes the case in which the old generation can borrow from the free market in the next period the amount of commodity that need to return is decreasing; no one can refund the debt so the price value is higher than fundamental value. As the illustration Figure 5 shows that with the existence of variable $b$, as the fraction of the debt that will not be refund, when $s'_1 > 0$ and $s_1 < 0$ the price market price values will increase along with the increasing of variable $b$. As for the case $s'_1 < 0$ and $s_1 > 0$, the bubble does exist but in smaller values as illustrate in Table 1.

4. Conclusion

In the samuelsonian three-period model young agent saving is always negative. That’s why the assets price values in competitive market will always equal to its fundamental value. In the three-period samuelsonian model the bubble does not exist because market price is equal to the fundamental value of the unique perishable goods. Samuelson leaves the open question about the consequence of anonymous transaction on the price value: “if the model end up with positive saving of young agents then the voluntary trade can be happened because not like the middle-aged the young willing to trade with anyone”. In this paper we establish the condition under which the young agent saving is positive. Under anonymous transaction we show the difference between market price and fundamental value of

Figure 5. Fundamental Value, Asset Price, and Asset Bubbles in Three-period OLG Heterogeneous Agents Model for case $s'_1 > 0$ and $s_1 < 0$
perishable goods. Our results show that the bubbles can exist even in economic with the perishable goods without production.

In heterogeneous agents model we assume that there are two types of agents who have different consumption preference. The first group has the logarithmic utility function like the samuelsonian model, and another group has linear-logarithmic utility function. Thus we prove that the part of young agents can become lenders, because we can find the situation where one group of young agents has positive saving. In this situation the transaction under anonymous market possible, then there is a possibility the old agent can borrow from the market. As a consequence the asset market price will be different with its fundamental value.

In Conclusion, we have proved the existence of endogenous bubble under anonymous rules of free market in heterogeneous agents model, where the market price is higher than fundamental value of the perishable goods.

References


